# A FRAMEWORK FOR THROUGHPUT AND STABILITY ANALYSIS OF A DS-CDMA NETWORK*

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Abstract - We propose a framework for throughput and stability analysis of a Direct Sequence Code Division Multiple Access (DS-CDMA) network. Given chip rate, processing gains, packet structure, coding scheme, and the characteristics of the wireless channel, we derive a system of non-linear fixed point equations for the sustainable link bit rates. The dimension of this system is equal to the number of network nodes receiving transmissions from other nodes. Generally, this non-linear system may have multiple locally stable solutions which indicates a possible instability of the DS-CDMA network because the persistent link bit rates may depend on the initial link loads. As an example, we consider throughput and stability of the reverse link in a cellular system, i.e., from mobiles to the base station. A reverse link is described by a single fixed point equation regardless of the number of mobiles.

## I. INTRODUCTION

The major obstacle for applying a methodology of queueing networks [1] to performance evaluation of a Direct Sequence Code Division Multiple Access (DS-CDMA) network lies in assigning bit service rates to the network links. In a DS-CDMA network transmission from a node $i$ to a node $j$ is seen as interference by transmissions from any node $n$ to any node $k$, such that $(n,k) \neq (i,j)$, and consequently, reduces the bit service rates for links $(n,k)$. This interaction introduces *positive feedback* into the system, i.e., temporary congestion in any part of the network caused by fluctuations in the incoming traffic or the wireless channel quality, reduces the network throughput around the congested area, which in turn increases the congestion, etc. This mechanism may be responsible for spreading local congestion into a larger portion of the network.

In this paper we propose a simple aggregated framework for analyzing these kinds of problems. Given chip rate, processing gains, packet structure, coding scheme, and characteristics of the wireless channel, the model allows one to estimate the sustainable link bit rates and assess the stability of a DS-CDMA network. The sustainable link bit rates are determined by the stable solutions of the system of non-linear fixed point equations. The number of equations is equal to the number of network nodes receiving transmissions from other nodes. This system has a unique globally stable solution under light and heavy traffic conditions. However, for moderate incoming traffic the corresponding system may have multiple locally stable solutions describing persistent lightly loaded and congested network modes. This phenomenon indicates a possible instability of the DS-CDMA network. The network stays in the lightly loaded or congested mode for a long time depending on the initial load. Note that this phenomenon is similar to the unstable behavior of some other multiple access algorithms, such as ALOHA [1]. The instability of a reverse link in DS-CDMA for data transmission was also discussed in [2].

The paper is organized as follows. Section 2 describes a queueing model of a link in a DS-CDMA network. Section 3 uses a simple model for calculating the packet error probability. Section 4 derives a system of fixed point equations that determine the link utilization in a DS-CDMA network. Section 5 analyses the throughput and stability of the reverse link: from mobiles to the base station. A reverse link is described by a single fixed point equation regardless of the number of mobiles.

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## II. QUEUEING MODEL OF DS-CDMA

Fig. 1 presents a queueing model of a link $(i, j)$ in a packet-based, DS-CDMA network.

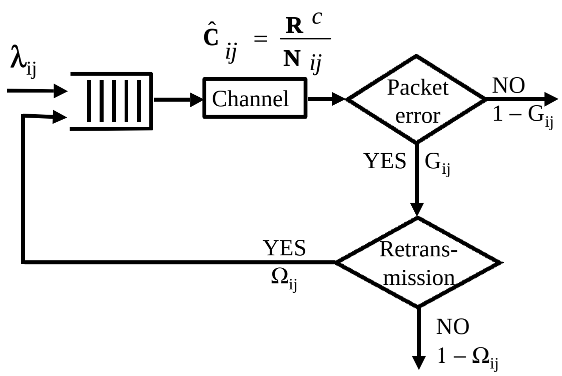



Fig. 1 Queueing model of DS-CDMA

For simplicity we assume that packets have fixed length- $L[bits]$. Node $i$ generates traffic to node $j$ at an average rate $l_{ij}[bits/\sec]$. This traffic enters a buffer of size $B_{ij}[packets]$, if buffer space is available. Traffic from the buffer is transmitted at the constant rate

$$\hat{c}_{ij} = \frac{R^c}{N_{ij}}, \tag{1}$$

where $R^c$ is the chip rate of the system and $N_{ij}$ is the processing gain for the transmission from node $i$ to node $j$. Transmission rate (1) is also the peak bit service rate provided by link $(i, j)$. Due to interference, only a fraction of packets transmitted from node $i$ to node $j$ is accepted at node $j$ as received correctly (i.e. without errors, or with an acceptable number of bit errors). These packets leave the system as successfully delivered. Let $1-G_{ij}$ be the probability that a transmitted packet is successfully delivered. Packet error probability $G_{ij}$ depends on the bit energy-to-noise ratio, coding scheme, etc. and is discussed in the next section. Some packets that are received with an unacceptable number of bit errors are retransmitted. We model this feature of the system assuming that each packet received with unacceptable number of bit errors is retransmitted with probability $\Omega_{ij}$ or leaves the system with probability $1-\Omega_{ij}$. We assume, for the purpose of computing throughputs, that $G_{ij}$ and $\Omega_{ij}$ remain constant regardless of how many times a packet is transmitted until it is either received with an acceptable number of errors or it leaves the system. We also assume that a packet stays in the buffer until it leaves the system.

The link queueing performance can be characterized by the probability that a packet is delivered successfully and by the probability distribution of the delay for successfully delivered packets. Note that delay depends on scheduling discipline for fresh and retransmitted packets. Due to retransmissions, performance evaluation of a link in presence of bursty traffic is not a simple task even if the probabilities $G_{ij}$ are fixed. The major problem is that retransmissions from node $i$ to node $j$ are jointly statistically dependent for different origin destination pairs $(i, j)$. This leads to a strong statistical correlation among different queues in a DS-CDMA network. It can be shown that temporary congestion, and consequently packet delays and buffer overflows tend to occur synchronously at different nodes of a DS-CDMA network. A performance analysis of the resulting collection of interacting systems shown on Fig. 1 is an unrealistic task. Note that attempts to analyze performance of each session separately assuming that packet error probabilities $G_{ij}$ are constant, in effect, eliminates statistical multiplexing and significantly underestimates the system ability to handle multimedia traffic.

This paper is limited to throughput analysis of a DS-CDMA network in the case of infinitely large buffers: $B_{ij} = \infty$. It is easy to see that given $G_{ij}$, the average clearing rate for traffic backlogged on a link $(i, j)$ is:

$$c_{ij} = \hat{c}_{ij}(1-\Omega_{ij}G_{ij}). \tag{2}$$

Since retransmissions introduce additional packet delays, for real-time traffic with very stringent delay requirements no retransmission is allowed ($\Omega_{ij} = 0$), and the clearing rate is constant and equal to the peak service rate: $c_{ij} = \hat{c}_{ij}$. For non real-time traffic with very loose delay requirements, there is no restrictions on the number of retransmissions ($\Omega_{ij} = 1$) and the average clearing rate is: $c_{ij} = \hat{c}_{ij}(1-G_{ij})$.

## III. PACKET ERROR PROBABILITY

The packet error probability $G_{ij}$ is a decreasing function of the bit energy-to-interference ratio $(E_b / I_0)_{ij}$:

(3) $$G_{ij} = G_{ij}[(E_b / I_0)_{ij}].$$

The bit energy-to-interference ratio is:

(4) $$(E_b / I_0)_{ij} = \frac{N_{ij} P_{ij} x_{ij}}{s_j^2 + \sum_{n \neq j} x_{nj} \sum_{k:(n,k)\neq(i,j)} P_{nk} d_{nk}},$$

where $P_{ij}$ is the transmission power from node $i$ to node $j$, $x_{ij}$ is the power attenuation factor from node $i$ to node $j$, $d_{nk} = 1$ if node $n$ transmits to node $k$ and $d_{nk} = 0$ otherwise, and $s_j^2$ is the average power of the background noise at node $j$.

The packet error probabilities $G_{ij}$ depend on the bit energy-to-interference ratio $(E_b / I_0)_{ij}$, as well as wireless channel quality, packet structure, and coding scheme. In this section we derive an expression for the packet error probability $G_{ij}$ under the following assumptions:

- perfect interleaving, i.e., all bit errors in the packet are jointly statistically independent,
- all bit errors are discovered, and
- a packet is accepted if the number of erroneous bits does not exceed threshold $l_{ij}^* = a_{ij} L$ where threshold $a_{ij}$ is a constant depending on the error correction coding scheme used at node $j$ for packets received from node $i$.

Under these assumptions, the number of bit errors in the packet is described by the binomial distribution [3] and the packet error probability is:

(5) $$G_{ij} = \sum_{l=l_{ij}^*+1}^{L} \frac{L!}{l!(L-l)!} g_{ij}^l (1-g_{ij})^{L-l},$$

where $g_{ij}$ is the bit error probability for a transmission from node $i$ to node $j$. Since typically a packet consists of a large number of bits, we can approximate the binomial distribution (5) by the normal distribution assuming that

(6) $$L \to \infty$$

and

(7) $$h_{ij} = \frac{a_{ij} - g_{ij}}{\sqrt{g_{ij}(1-g_{ij})}} L_{ij}^{1/2}$$

is constant. Under (6)-(7) the packet error probability (5) takes the following form [3]:

(8) $$G_{ij} = \frac{1}{\sqrt{2p}} \int_{h_{ij}}^{\infty} e^{-\frac{x^2}{2}} dx.$$

Typically, the packet error probability is small, and formula (8) can be simplified further [3]:

$$G_{ij} \approx \frac{1}{\sqrt{2p}} \frac{e^{-\frac{h_{ij}^2}{2}}}{h_{ij}} \quad \text{as } h_{ij} \to \infty.$$

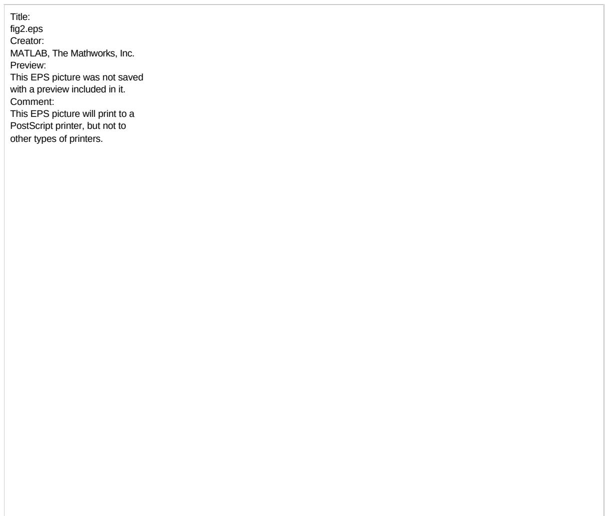
The bit error probability $g_{ij}$ is a decreasing function of the bit energy-to-interference ratio $(E_b / I_0)_{ij}$:

(9) $$g_{ij} = g_{ij}[(E_b / I_0)_{ij}].$$

It is usually assumed that

(10) $$g_{ij} = e^{-b_{ij}[(E_b / I_0)_{ij}]}$$

where parameter $b$ characterizes the asymptotic coding gain. Fig. 2 shows the normalized average clearing rate $c_{ij} / \hat{c}_{ij}$ as a function of the bit energy-to-interference ratio $(E_b / I_0)$ for different retransmission probabilities $\Omega = \Omega_1 < \Omega_2 < \Omega_3$.




## IV. FIXED POINT EQUATIONS FOR THE LINK UTILIZATION

Formula (2) gives the average clearing rate for a link $(i, j)$ as a function of the bit energy-to-interference ratio $(E_b / I_0)_{ij}$: $c_{ij} = c_{ij}[(E_b / I_0)_{ij}]$. The average utilization of link $(i, j)$ is:

$$(11) \qquad r_{ij} = \min\{1, \frac{l_{ij}}{\mathrm{E}\{c_{ij}[(E_b / I_0)_{ij}]\}}\},$$

where $\mathrm{E}\{.\}$ denotes mathematical expectation. In this section we derive approximate fixed point equations for the average link utilization based on the following approximations. First, we assume that the clearing rate averaged over bit energy-to-interference ratio can be approximately replaced by the clearing rate as a function of the average bit energy-to-interference ratio:

$$(12) \qquad r_{ij} \approx \tilde{r}_{ij} = \min\{1, \frac{l_{ij}}{c_{ij}[\mathrm{E}\{(E_b / I_o)_{ij}\}]}\}.$$

Obviously,

$$(E_b / I_0)_{ij} \geq (E_b / I_o)^*_{ij}$$

where

$$(E_b / I_o)^*_{ij} = \frac{N_{ij} P_{ij} x_{ij}}{s_j^2 + \sum_{n \neq j} x_{nj} \sum_{k \neq n} P_{nk} d_{nk}}.$$

We also make the following two approximations:

$$(13) \qquad \mathrm{E}\{(E_b / I_0)_{ij}\} \approx \mathrm{E}\{(E_b / I_o)^*_{ij}\}$$

$$(14) \qquad \mathrm{E}\{(E_b / I_o)^*_{ij}\} \approx \overline{(E_b / I_o)}_{ij},$$

where

$$(15) \qquad \overline{(E_b / I_o)}_{ij} = \frac{N_{ij} P_{ij} x_{ij}}{s_j^2 + \sum_{n \neq j} x_{nj} \sum_{k \neq n} P_{nk} \mathrm{E}\{d_{nk}\}}.$$

It can be shown that all three approximations (12)-(14) are valid for large networks where each received signal is a superposition of transmissions by a large number of nodes. Combining (12)-(14) and taking into account that

$$(16) \qquad \mathrm{E}\{d_{nk}\} = r_{nk},$$

we obtain the following system of equations:

$$(17) \qquad \tilde{r}_{ij} = \min\{1, \frac{l_{ij} N_{ij}}{R^c (1 - \Omega_{ij} \tilde{G}_{ij})}\}$$

$$(18) \qquad \tilde{G}_{ij} = G_{ij}[\overline{(E_b / I_o)}_{ij}]$$

$$(19) \qquad \overline{(E_b / I_o)}_{ij} = \frac{N_{ij} P_{ij}}{s_j^2 + \sum_{n \neq j} x_{nj} \sum_{k \neq n} P_{nk} \tilde{r}_{nk}}.$$

Substituting expression (19) for $\overline{(E_b / I_o)}_{ij}$ into the right-hand side of (18), and then substituting the resulting expression (18) for $\tilde{G}_{ij}$ into the right-hand side of (17), we obtain a system of $M$ fixed point equations for $M$ average link utilization $\tilde{r}_{ij}$, where $M$ is the number of links in the network. It is possible to reduce the dimension $M$ of this system. Let us introduce the average power $P_k^+$ received by node $k$ as a result of transmissions by all other nodes $n \neq k$:

$$(20) \qquad P_k^+ = \sum_{n \neq k} x_{nk} \sum_{j \neq n} P_{nj} \tilde{r}_{nj},$$

and rewrite equations (17)-(19) as follows:

$$(21) \qquad P_k^+ = \sum_{n \neq k} x_{nk} \sum_{j \neq n} P_{nj} \min\{1, \frac{l_{nj} N_{nj}}{R^c (1 - \Omega_{nj} \tilde{G}_{nj})}\}$$

$$(22) \qquad \tilde{G}_{nj} = G_{nj}(\frac{N_{nj} P_{nj} x_{nj}}{s_j^2 + P_j^+}),$$

forming a system of $K$ fixed point equations for $K$ unknowns (20), where $K$ is the number of nodes in the network receiving transmissions from other nodes. Once this system is solved, the average link utilization $\tilde{r}_{ij}$ can be calculated using (12). Note that typically $K << M$ and it is much easier to solve the system (21)-(22) than the system (17)-(19). In the next section we consider a case of reverse link from mobiles to the base station where $K = 1$.

## V. THROUGHPUT AND STABILITY OF THE REVERSE LINK

In this section we consider the reverse link in a DS-CDMA network, where $J$ mobiles transmit to the base station. The average power received at the base station from all these transmissions is:

$$(23) \qquad P^+ = \sum_j x_j P_j \min\{1, \frac{l_j N_j}{R^c (1 - \Omega_j \tilde{G}_j)}\},$$

where $P_j$ is the power transmitted by mobile $j$, $x_j$ is the power attenuation factor from mobile $j$ to the base station, $\Omega_j$ is the retransmission probability for mobile $j$, and

$$(24) \qquad \tilde{G}_j = G_j(\frac{N_j P_j x_j}{s^2 + P^+})$$

is the packet error probability for mobile $j$. Substituting (24) into the right-hand side of (23), we obtain a single equation for the average power received at the base station from all mobiles $P^+$:

$$(25) \qquad P^+ = j(P^+)$$

regardless of the number of mobiles $J$. After solving equation (25), the average uplink utilization from mobile $j$ to the base station can be estimated as follows:

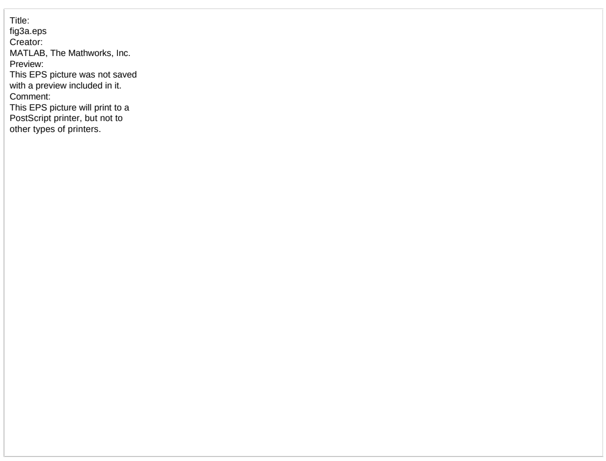
$$(26) \qquad r_j \approx \tilde{r}_j = \min\{1, \frac{l_j N_j}{R^c(1-\Omega_j \tilde{G}_j)}\}.$$

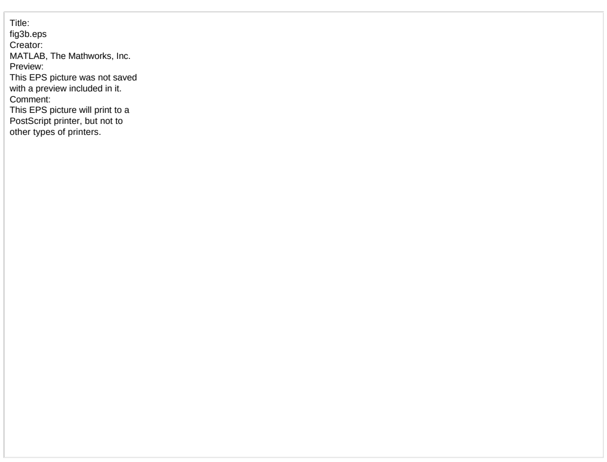
In the particular case of $J$ identical mobiles transmitting data traffic with very loose delay requirements: $\Omega_j = 1$, $x_j = x$, $P_j = P$, $l_j = l$, $N_j = N$, $G_j = G$, equations (23)-(24) reduce to the following system:

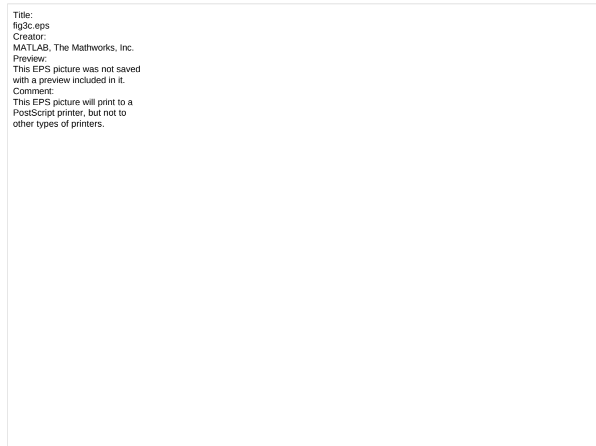
$$(27) \qquad \tilde{r} = \min\{1, \frac{q}{1-\tilde{G}}\}$$

$$(28) \qquad \tilde{G} = G(\frac{N/J}{s_{eff}^2 + \tilde{r}}),$$

where $q = \frac{Nl}{R^c}$ and $s_{eff}^2 = \frac{s^2}{JPx}$




Substituting (28) into the right-hand side of (27) we obtain a following fixed point equation:

$$(29) \qquad \tilde{r} = f(\tilde{r})$$

Solution of this equation for light, moderate and heavy incoming traffic is shown on Fig. 3a, 3b and 3c respectively. For light (heavy) incoming traffic equation (29) has a unique globally stable solution $\tilde{r}^* < 1$ ($\tilde{r}^* = 1$) shown by thick dots on Fig.3. For moderate incoming traffic these two solutions coexist as locally stable. This indicates bi-stability of the system. Note that similar type of bi-stability has been observed in large-scale wired networks [4].

## VI. ACKNOWLEDGEMENT

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